BIFURCATION TO PEAR-SHAPED EQUILIBRIA OF PRESSURIZED SPHERICAL MEMBRANES

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Abstract—We study the problem of non-spherical, axisymmetric equilibria of an inflated, spherical membrane. We model the membrane as a two-dimensional elastic body characterized by a general class of strain-energy functions, and we consider a general class of loading devices, including (soft) pressure control and (hard) control of the total mass of gas enclosed by the membrane as special cases. Employing tools of modern bifurcation theory, we illuminate the precise necessary and sufficient conditions for bifurcation from the spherical state to an axisymmetric "pear-shaped" state, and we perform a local post-bifurcation analysis. In particular, for a large class of physically reasonable strain-energy functions, we demonstrate the existence of an isola bifurcation (closed loop of non-spherical solutions), which is consistent with experimental observations.

1. INTRODUCTION

Inflation of an initially spherical membrane under internal pressure has been studied by several investigators. Of particular interest is the onset of non-spherical deformations, which have been observed in experiments. Fedos'ev [1] studied the axisymmetric solutions of the linearized equations of equilibrium employing a particular strain-energy function proposed by Biderman [2]. Fedos'ev found that the linearized equilibrium equations lead to the classical Legendre equation, from which the bifurcation pressure (at the onset of non-spherical response) can be calculated.

Alexander [3] examined the so-called tensile instability of spherical deformations for a particular strain-energy function, which he proposed himself. Basically, he considered instability with respect to spherical disturbances only, and consequently, could not detect bifurcation to non-spherical shapes. His analysis is based upon the assumption that the internal pressure is controlled during the inflation, which is usually not the case, as he pointed out himself [3]. Alexander's paper contains some interesting photographs showing the occurrence of non-spherical deformations of an initially spherical neoprene balloon. The balloon appears to be spherical until \( r = 1.6 \) (\( r \) being the ratio of the deformed radius to the undeformed radius). At this point, a "pear-shaped" equilibrium is observed, with the balloon thickness greater in one hemisphere than in the other. Moreover, the spherical deformation is reported to be regained at about \( r = 4.3 \).

As is well known, the existence of non-trivial solutions of the linearized equations does not necessarily imply the existence of bifurcating solution branches, the analysis of which is certainly desirable. Needleman [4] approached this problem numerically for a strain-energy function proposed by Ogden [5]. The existence of bifurcating branches was not justified directly in Needleman's paper, where he introduced an initial imperfection in order to carry out a numerical analysis by means of the Ritz-Galerkin method. Also, the volume enclosed by the deformed membrane was treated as the control variable or bifurcation parameter, which does not appear to be reasonable from a physical point of view. Another numerical analysis of this problem for an Ogden material (without initial imperfections) was carried out by Haughton [6], with internal pressure as the control variable. His results indicate a "closed-loop" or isola bifurcation, i.e., a branch of (non-spherical) solutions connecting two bifurcation points.

The existing literature on non-spherical deformations of initially spherical membranes seems to be incomplete in at least two aspects. First, analytic results seem to be available only for the linearized equations, i.e., only necessary conditions for bifurcation have been...
obtained. While a numerical analysis can deliver much useful information, it is somewhat ineffective in gaining general insight into the problem. In particular, numerical computations can only be carried out for specific choices of the strain-energy function. Therefore, general relations between the qualitative features of the solutions and the properties of the strain-energy function are not likely to be apparent. Secondly, the choice of the bifurcation parameter in past studies appears to have been dictated by mathematical convenience rather than by physical relevance. Indeed, such experiments are usually accomplished by controlling the mass of the gas enclosed by the deformed membrane.

In this paper, we study the bifurcation problem for a general class of physically reasonable strain-energy functions. The basic equations are derived in Section 2. The problem is formulated as a minimization of the total energy functional, which comprises the strain-energy stored in the membrane and the potential energy of the loading device. The latter is assumed to depend on the volume enclosed by the deformed membrane and a control variable in a general way. The spherical (trivial) solutions and their stabilities are studied in Section 3. With regard to spherical solutions, we show that the loading device corresponding to internal-pressure control is the most unstable. In Section 4, we obtain rigorous, local bifurcation results, corresponding to pear-shaped equilibria, for the non-linear problem. The general form of the bifurcation equation is computed and found to correspond to a pitch-fork bifurcation. Also, the spherical solution branch is shown to lose stability whenever non-spherical solutions occur. Finally, in Section 5, we consider the case in which the total mass of the gas enclosed by the membrane is controlled. For a more specific, but physically relevant, class of strain-energy functions, we establish the existence of an isolated bifurcation corresponding to pear-shaped equilibria.

2. FORMULATION

Consider an initially spherical elastic membrane, having unit radius in a natural state, subjected to an internal pressure. If the deformed shape of the membrane is presumed to be a smooth surface of revolution, then the deformation is completely determined by the position vector field of material points lying in any half-plane passing through the axis of symmetry. We introduce a rectangular coordinate system in such a plane, with origin located at the centre of the undeformed membrane. The \( x_1 \)-axis is taken to coincide with the axis of symmetry. Let \( (x_1(\theta), x_2(\theta)) \) denote the spatial coordinates of the material point having polar coordinates \( r = 1 \) and \( \theta \) in the undeformed configuration, as depicted in Fig. 1.

An axisymmetric deformation can then be described by two scalar-valued functions \( x_1(\theta) \) and \( x_2(\theta), \theta \in [0, \pi] \). The functions \( x_1 \) and \( x_2 \) are assumed to be of class \( C^2 \) and satisfy the conditions

\[
x_1(0) = x_1(\pi) = x_2(0) = x_2(\pi) = 0, \tag{1}
\]

and

\[
x_1\left(\frac{\pi}{2}\right) = 0, \tag{2}
\]

where prime denotes differentiation with respect to \( \theta \). The boundary conditions (1) ensure

![Fig. 1. Generator of a typical axisymmetric deformation.](image-url)
that the deformed membrane is a smooth, closed surface, while condition (2) is imposed to exclude rigid horizontal translations. Let \( \lambda_1 \) and \( \lambda_2 \) denote the principal stretches. A straightforward calculation shows that the principal directions coincide with the longitudinal and latitudinal directions, and

\[
\lambda_1 = \frac{x_2}{\sin \theta}, \quad \lambda_2 = (x_1'^2 + x_2'^2)^{1/2}.
\]

We shall employ an energy criterion to study the stability of equilibrium configurations. The membrane and the loading device are treated as an isolated system associated with a total potential energy \( E \) consisting of two parts: the strain energy \( E_s \) stored in the membrane and the potential energy \( E_p \) associated with the loading device. Stable equilibria are characterized as minima of the total energy in a suitably chosen subset of admissible deformations.

The elastic membrane is assumed to be homogeneous and isotropic in its undeformed state. The strain energy density (per unit undeformed area) is given by a sufficiently smooth (\( C^2 \) is enough) strain-energy function \( W(\lambda_1, \lambda_2) \) of the principal stretches. By isotropy, \( W \) is symmetric in its two arguments:

\[
W(\lambda_1, \lambda_2) = W(\lambda_2, \lambda_1).
\]

Since the reference configuration is a natural state, we have

\[
W(1, 1) = 0, \quad i = 1, 2,
\]

where the subscript \( i \) denotes partial differentiation with respect to \( \lambda_i \). The strain energy in the membrane is then given by

\[
E_s = 2\pi \int_0^\pi W(\lambda_1, \lambda_2) \sin \theta \, d\theta.
\]

We assume that the potential energy \( E_p \) associated with the loading device is given by a \( C^2 \) potential function \( \phi \) of the volume \( V \) enclosed by the deformed membrane and a control variable \( \mu \in \mathbb{R}^+ \):

\[
E_p = \phi(V, \mu),
\]

where

\[
V = -\pi \int_0^\pi x_1' x_2' \, d\theta.
\]

The internal pressure \( p \) is then given by

\[
p = -\varphi(V, \mu),
\]

where a subscript denotes partial differentiation with respect to that argument. We shall consider the case where \( p \) is non-negative. Our model is quite general, covering a number of common cases. For example, if

\[
\varphi(V, \mu) = -\mu V,
\]

then \( \mu = p \), i.e. the internal pressure is prescribed. On the other hand, if the membrane is inflated by a specified amount of gas, then the potential energy \( \phi \) is the Helmholtz free energy of the gas, which, for an ideal gas, has the form

\[
\varphi(V, \mu) = -k \mu \ln \frac{V}{\mu},
\]

where \( k \) is a positive constant of the gas, and \( V \) has been properly normalized. In this case, the control variable \( \mu \) represents the total mass of the enclosed gas.

The total energy is given by

\[
E = E_s + E_p.
\]

By the energy stability criterion, a stable equilibrium deformation minimizes the above total energy. The first variation condition of this minimization problem leads to the following
equations of equilibrium:

\[ p x_2 x_2' + \left( \frac{x_1' W_2 \sin \theta}{\lambda_2} \right)' = 0, \quad p x_1' x_2 + W_1 - \left( \frac{x_1' W_2 \sin \theta}{\lambda_2} \right)' = 0, \]  

where \( p \) is given by (4). Note that (7) is a system of integro-differential equations (since \( p \) depends on \( x_1 \) and \( x_2 \) through (3) and (4)). Equations (7) can be reduced to

\[ p x_2' \lambda_2 + 2 x_1' W_2 \sin \theta = 0, \quad x_2' W_1' - \lambda_2 (W_2' \sin \theta) = 0. \]  

Equation (8) is the first integral of (7), the constant of integration having been determined by the boundary conditions (1).

3. SPHERICAL SOLUTIONS

A deformation is spherical if

\[ x_1(\theta) = \lambda \cos \theta, \quad x_2(\theta) = \lambda \sin \theta, \]  

where \( \lambda > 0 \) corresponds to the radius of the deformed membrane and to the equal principal stretches of the deformation. We find it convenient to define

\[ \tilde{W}(\lambda) \equiv W(\lambda, \lambda). \]  

By the symmetry of the strain-energy function, the deformation (9) satisfies the equations of equilibrium (8) if and only if

\[ p = \tilde{W}'(\lambda)/\lambda^2, \]  

where \( p \) is given by (4) with its argument \( V \) evaluated at \( 4\pi \lambda^3/3 \). Henceforth, unless stated otherwise, the potential function \( \Phi \) and its derivatives are evaluated at \( (4\pi \lambda^3/3, \mu) \), and the strain-energy function \( \tilde{W} \) and its derivatives are evaluated at \( (\lambda, \lambda) \), with \( \lambda \) satisfying (11).

Of course, we can obtain (11) in an alternative way. For spherical deformations (9), the total energy is given by

\[ E_0 = 4\pi \tilde{W}(\lambda) + \Phi \left( \frac{4\pi \lambda^3}{3}, \mu \right). \]  

If the spherical deformation is in equilibrium, then the radius \( \lambda \) must be a stationary point of the right-hand side of (12). Equation (11) then follows. Moreover, if a solution of (11) corresponds to a stable equilibrium, then it is a local minimum of \( E \) and hence, of \( E_0 \) as well. Therefore, the condition

\[ -2 \tilde{W}'/\lambda + \tilde{W}'' + 4\pi \lambda^4 \phi_{\Phi V V} \geq 0 \]  

is necessary for stability.

If pressure is controlled, then by (5), we have \( \phi_{\Phi V V} = 0 \), and (13) reduces to

\[ -2 \tilde{W}'/\lambda + \tilde{W}'' \geq 0. \]  

Recall that (11) gives the internal pressure \( p \) needed to maintain the spherical deformation (9) as a function of the radius \( \lambda \). Inequality (14) then requires this function to be non-decreasing. More generally, it is physically reasonable to assume that \( \phi_{\Phi V V} \geq 0 \), which implies that the pressure \( p \) is non-increasing in the volume \( V \), cf. (4). Thus, inequality (14) implies inequality (13), and we conclude that pressure control is the most unstable. For example, if \( \phi \) is of the form (6), then \( \phi_{\Phi V V} = k\mu/V^2 \) and inequality (13) becomes

\[ \tilde{W}'/\lambda + \tilde{W}'' \geq 0, \]  

which is obviously weaker than (14). As a result, a spherical deformation which is unstable during pressure control may be stable when the mass of the enclosed gas is controlled. In the latter case, as the mass of the gas is increased, the internal pressure may decrease accompanied by an increase of the radius.

While inequality (13) is certainly necessary for the spherical equilibrium to be stable, it is not sufficient, since it was derived for spherical disturbances only. Indeed, as we shall see below, a spherical deformation satisfying (13) may become unstable and give rise to bifurcating non-spherical deformations.
4. BIFURCATION ANALYSIS

It is convenient in a bifurcation analysis to reformulate the equations of equilibrium in an abstract way. Let \( \mathcal{A} \) denote the set of axisymmetric deformations:

\[
\mathcal{A} = \left\{ x \in C^2([0, \pi]; \mathbb{R}^2) : \ x'_1(0) = x'_1(\pi) = x'_2(0) = x'_2(\pi) = x_1 \left( \frac{\pi}{2} \right) = 0 \right\},
\]

where \( x \equiv (x_1, x_2) \). We identify the left side of (8) with a mapping

\[
G : \mathcal{A} \times \mathbb{R} \to C^0([0, \pi]; \mathbb{R}^2) \text{ via }
\]

\[
G(x, \mu) = (-\varphi_x x'_2 \lambda_2 + 2x'_1 W_2 \sin \theta, x'_2 W_1 - \lambda_2 W_2 \sin \theta').
\]

The smoothness of \( \varphi \) and \( W \) insures that \( G \) is Fréchet differentiable. The equations of equilibrium (8) along with the boundary conditions (1) and (2) can now be expressed succinctly as

\[
G(x, \mu) = 0, \quad (x, \mu) \in \mathcal{A} \times \mathbb{R} + .
\]

Suppose that equation (17) has a spherical solution branch

\[
x(\mu) = (\lambda(\mu) \cos \theta, \lambda(\mu) \sin \theta), \quad \mu \in I,
\]

where \( I \) is a subinterval of \( \mathbb{R} + \), and \( \lambda(\mu) > 0 \) satisfies (11), namely,

\[
-\varphi_x \left( \frac{4\pi \lambda^5(\mu)}{3} \right) = W'(\lambda(\mu)) \lambda^2(\mu).
\]

Let \( u \) be a small axisymmetric displacement superposed on the spherical deformation (18):

\[
u = (v_1 \cos \theta - v_2 \sin \theta, v_1 \sin \theta + v_2 \cos \theta).
\]

Here we have introduced, for the convenience of further calculation, the normal and tangential components \( v_1 \) and \( v_2 \), respectively, of \( u \) relative to the spherical configuration. The linearized equations of equilibrium about the spherical deformation (18) are given by

\[
A(\mu) u = 0,
\]

where \( A(\mu) = G_s(x(\mu), \mu) : \mathcal{A} \to C^0([0, \pi]; \mathbb{R}^2), G_s \) denoting the Fréchet derivative of \( G \) with respect to \( x \). A straightforward calculation yields (21) explicitly:

\[
W_1(v'_1 \cot \theta + 2v_1 + v_2 \cot \theta) - \lambda W_1(\lambda(v_1 + v_2)) - \lambda W_1(\lambda(v_1 + v_2 \cot \theta))
\]

\[
- \pi \lambda^5 \varphi_{vv} \int_0^\pi v_1 \sin \theta \, d\theta = 0,
\]

\[
W_1(v'_1 - v_2) - \lambda W_1(v'_2 + v_1 + v_2 \cot \theta - v_2 \cot^2 \theta) - \lambda W_1(v'_1 - v_2) = 0.
\]

It also follows from (1), (2) and (20) that

\[
v_1'(0) = v_1'(\pi) = v_2'(0) = v_2'(\pi) = v_2 \left( \frac{\pi}{2} \right) = 0.
\]

Remark. Our linearized equations generalize Feodos'ev's results [1] in two ways. First, equations (22) are valid for a general form of the strain-energy function. Moreover, Feodos'ev's analysis presumes pressure control, for which \( \varphi_{vv} = 0 \), cf. (5). In our formulation, a displacement may be accompanied by a change of the internal pressure as indicated by the emergence of the last term on the left-hand side of (22). Incidentally, that extra term renders (22), an integro-differential equation.

A necessary condition for bifurcation is that (21) admit non-trivial solution pairs \( (u, \mu) \neq (0, \mu) \), i.e. we seek values of \( \mu \) such that \( A(\mu) \) is not invertible. Indeed, if \( A(\mu) \) is invertible, then the implicit function theorem insures the existence of a unique local branch of solutions, namely, \( \{(x(\mu), \mu) : \mu \in I\} \). Equations (22) are related to the classical Legendre equation. To see this, we solve (22) for \( v_2 \) and substitute it into (22) to find that

\[
\lambda W_1(v'_2 + v'_1 + v_2 \cot \theta) + (W_1 + \lambda W_1 - \lambda W_1 - \lambda W_1 - \lambda W_1 - \lambda W_1 - \lambda W_1) v_1
\]

\[
= C(W_1 - \lambda W_1 - \lambda W_1) \cos \theta + \pi \lambda^5 (W_1 + \lambda W_1 - \lambda W_1) \varphi_{vv} \int_0^\pi v_1 \sin \theta \, d\theta,
\]

(24)
where \( C \) is a constant of integration. If \( W_1 W_{11} = 0 \), the solution of (24) is immediate. If \( W_1 W_{11} \neq 0 \), making the change of variables \( s = \cos \theta \) in (24) yields

\[
\lambda W_1 W_{11} \left[ (1 - s^2) \frac{d^2 v_1}{ds^2} - 2s \frac{dv_1}{ds} \right] + (W_1 + \lambda W_{11} - \lambda W_{12}) (2W_1 - \lambda W_{11} - \lambda W_{12}) v_1
\]

\[
= C(W_1 - \lambda W_{11} - \lambda W_{12}) s + \pi \lambda^2 (W_1 + \lambda W_{11} - \lambda W_{12}) \phi_{vv} \int_{-1}^{1} v_1 ds,
\]

which is recognized as an inhomogeneous Legendre equation whose solution is the sum of a polynomial and the solution of the corresponding homogeneous Legendre equation.

As is well-known, the bounded solutions of the Legendre equation on \([-1, 1]\) are the Legendre polynomials. Hence, the solution of (24) is of the following form

\[
v_1 = D P_n(\cos \theta) + A \cos \theta + B,
\]

where \( A, B \) and \( D \) are constants, and \( P_n \) the Legendre polynomial of order \( n \)

\[
P_n(s) = \frac{1}{2^n n!} \frac{d^n}{ds^n} (s^2 - 1)^n.
\]

For an integer \( n \), substitution of (25) into (22) gives \( v_2 \) and the relation between \( W_1, W_{11}, W_{12} \) and \( \omega_{vv} \) ensuring that such \( v_1 \) and \( v_2 \) are indeed solutions of (22). Of course the constants in (25) must be chosen to satisfy (22) and (23). Since

\[
P_0(\cos \theta) = 1, \quad P_1(\cos \theta) = \cos \theta,
\]

we can choose \( D = 0 \) for the cases \( n = 0 \) and \( n = 1 \).

The "mode-zero" solution \((n = 0)\)

\[
v_1 = 1, \quad v_2 = 0,
\]

is valid when

\[
2 \dddot{\bar{W}} - \lambda \dddot{\bar{W}} - 4 \pi \lambda^3 \phi_{vv} = 0.
\]

Comparing the above equation with (13), we see that the mode-zero solution exists precisely at a point where the spherical deformation is about to lose stability relative to spherical disturbances, i.e. at a limit point of the pressure vs stretch diagram, cf. (11).

The "mode-one" solution \((n = 1)\)

\[
v_1 = \cos \theta, \quad v_2 = 0,
\]

is valid when

\[
\dddot{\bar{W}} - \lambda \dddot{\bar{W}} = 0,
\]

regardless of the form of \( \phi \). Of course, \( \dddot{\bar{W}} > 0 \) for \( \lambda > 0 \) is physically relevant, in which case equation (28) satisfies (15) by equality but violates (14). Thus, a spherical deformation about which the linearized equilibrium equation possesses a mode-one solution is unstable when the internal pressure is controlled, but may be stable when the mass of the enclosed gas is controlled.

The solution (27) suggests a deformation where the two principal stretches are monotone increasing from one pole to the other, i.e. a pear-shaped equilibrium, agreeing with Alexander's experimental observation [3]. In the remainder of this paper, we consider bifurcation associated with the mode-one solution (27) only.

Suppose that \((x_0, \mu_0) = (\lambda(\mu_0) \cos \theta, \lambda(\mu_0) \sin \theta, \mu_0)\) is a candidate for a mode-one bifurcation point, i.e. (19) and (28) are satisfied at \( \lambda_0 = \lambda(\mu_0) \). It then remains to characterize the kernel and range of \( A_0 \equiv A(\mu_0) \) in order to perform a rigorous analysis of bifurcation.

**Lemma 1.** Suppose that

\[
W_1^0 \neq 0, \quad W_{11}^0 \neq 0, \quad \text{and} \quad W_1^0 - 2 \pi \lambda_0^2 \phi_{vv}^0 \neq 0,
\]

where \( W_1^0 \equiv W_1(\lambda_0, \mu_0) \), etc. Then the kernel and co-kernel of \( A_0 \) each have dimension one, and are spanned by \( \phi(\theta) = (\cos^2 \theta, \sin \theta \cos \theta) \) and \( \psi(\theta) = (0, \sin \theta) \), respectively, relative to the standard \( L^2 \) inner product.
Proof. Our previous calculations imply that \( \phi \) is a null vector of \( A^0 \), and it is straightforward to demonstrate that \( \phi \) is the only linearly independent solution of (21) at \( \mu = \mu_0 \). To show the property of the co-kernel of \( A^0 \), we first note that

\[
\int_0^{\pi} (0, \sin \theta) \cdot A^0 u \, d\theta = 0 \quad \forall u \in \mathcal{H},
\]

i.e., \( \psi(\theta) \) is contained in the co-kernel of \( A^0 \). Moreover, if \( w \in C^0([0, \pi]; \mathbb{R}^2) \) satisfies

\[
\int_0^{\pi} (0, \sin \theta) \cdot w \, d\theta = \int_0^{\pi} w_2 \sin \theta \, d\theta = 0,
\]

then the equation

\[
A^0 u = w
\]

has a solution of the form (20) with \( v_1 \) and \( v_2 \) given by

\[
v_1 = C + \cos \theta \int_{0}^{\pi} \frac{w_1 - 2x_1}{2 W_1^0 \sin \theta \cos^2 \theta} \, d\theta, \quad v_2 = -\sin \theta \int_{0}^{\pi} \frac{z}{w_2} \lambda_0 \frac{W_1^0 \sin^2 \theta}{W_1^0} \, d\theta,
\]

where

\[
z(\theta) = \int_{0}^{\pi} w_2 \sin \theta \, d\theta,
\]

and \( C \) is a constant such that

\[
C = \frac{\pi \lambda_0 \phi_{vv}}{W_1^0} \int_{0}^{\pi} v_1 \sin \theta \, d\theta.
\]

Lemma 1 insures that our bifurcation problem (17) can be reduced, via the well-known Liapunov–Schmidt method (e.g., cf. [7]), to the following standard one-dimensional problem

\[
g(x, y) = 0,
\]

where \( g : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}, \gamma \equiv \mu - \mu_0, \) and \( y \) is the component of the displacement \( x - x_0 \) in the direction of \( \phi \). Moreover, \( g \) is smooth in \( \Omega \) and satisfies the following conditions:

\[
g(-x, y) = -g(x, y),
\]

and

\[
g_{\gamma}^0 = g_{\gamma}^0 = 0,
\]

\[
\frac{\partial^2 g^0}{\partial y^{2k}} = 0, \quad k \text{ integer},
\]

\[
x_0 \equiv g_{\gamma \gamma}^0 = \frac{2\lambda_0 \phi_{yy}^0 (W_{11}^0 + 3W_{12}^0)}{3(W_1^0 - 2\pi \lambda_0 \phi_{yy}^0)},
\]

where \( g^0 \equiv g(0, 0), \) etc. The derivation of (31)–(33) is outlined in the Appendix.

A function \( g \) satisfying (32) is said to have \( Z_2 \)-symmetry. Such a function can be written, by Whitney [8], as

\[
g(x, y) = y h(y^2, y),
\]

for some smooth function \( h \).

The local bifurcation diagram of (31) can be classified by well-known singularity theory. The proof of the next theorem, which states a sufficient condition for bifurcation, can be found in Golubitsky and Schaeffer [7, VI. 2].

**Theorem 1.** Suppose that

\[
x_0 \neq 0, \quad g_{\gamma}^0 = g_{\gamma \gamma}^0 = \ldots = \frac{\partial^{2k-1} g^0}{\partial y^{2k-1}} = 0, \quad \beta_0 \equiv \frac{\partial^{2k+1} g^0}{\partial y^{2k+1}} \neq 0.
\]

Then the local bifurcation diagram of (31) (and hence, of (17)) near the origin is equivalent to the one generated by the normal form

\[
y(y^{2k} + \varepsilon y) = 0.
\]
where 
\[ \varepsilon \equiv \text{sgn} \, \alpha_0 \beta_0. \]

Equation (34) represents a pitch-fork bifurcation. The solution branch given by \( y = 0 \) corresponds to the spherical solution branch (18), while the solution branch given by \( y^{2k} + \varepsilon y = 0 \) corresponds to the non-spherical solution branch characterized by the mode one solution (27) of the linearized equations of equilibrium.

The experimental observations of the bifurcation from the spherical deformation to the pear-shaped deformation by Alexander [3] suggests that the spherical solution branch loses stability when the pear-shaped solutions occur. We thus examine the stability of the spherical solution branch under a displacement corresponding to the null vector (buckling mode) \( \Phi \).

The second variation of the total potential energy \( E \) evaluated at the spherical solution (9) is given by

\[
\delta^2 E = 2\pi \int_0^n \left\{ \frac{W_1 \sin \theta}{\lambda} \left[ 4u_1' u_2 - 2u_2^2 + (u_1' \cos \theta + u_2' \sin \theta)^2 \right] \\
+ W_{11} \left[ \frac{u_2^2}{\sin \theta} + (u_1' \sin \theta - u_2' \cos \theta)^2 \right] + 2W_{12} (- u_1' \sin \theta + u_2' \cos \theta) u_2 \right\} d\theta \\
+ \pi \lambda^2 \varphi_{\nu \nu} \left[ \int_0^n (u_1' - u_2) \sin^2 \theta d\theta \right]^2,
\]

which should be non-negative for all smooth functions \( u_1 \) and \( u_2 \) for stability. In particular, for \( (u_1, u_2) = \Phi \), we find that the spherical deformation (9) is stable only if

\[ q(\lambda) = -\dot{W}''(\lambda) + \lambda \ddot{W}''(\lambda) \geq 0. \quad (35) \]

Upon comparing (28) with (35) we see that \( q(\lambda) \) vanishes at the bifurcation point, hinting that \( q \) changes sign there.

**Theorem 2.** Assume the hypotheses of theorem 1. If \( \alpha_0 < (>) 0 \), then the spherical solution branch is unstable for \( \mu > (\mu) \).

**Proof.** From (18), (19) and (35), we find that

\[
\frac{d}{d\mu} q(\lambda(\mu)) = \frac{\lambda^2 \dddot{W}''(\Phi) \varphi_{\nu \nu}}{2\dot{W}'' - \lambda \ddot{W}'' - 4\pi \lambda^5 \varphi_{\nu \nu}},
\]

which by (28) and (33), is of the same sign as \( \alpha_0 \) at the bifurcation point. Hence, \( q(\lambda(\mu)) \) is monotone strictly increasing (decreasing) in \( \mu \) if \( \alpha_0 < (>) 0 \). Moreover, by (28), \( q(\lambda(\mu)) \) vanishes at the bifurcation point where \( \mu = \mu_0 \). The conclusion then follows.

**Corollary 1.** Assume \( \beta_0 > 0 \) and the hypotheses of theorem 1. Then the spherical deformation loses stability as soon as the pear-shaped deformation becomes possible.

**Proof.** By the assumptions, the bifurcation equation (34) becomes

\[ y(y^{2k} + \varepsilon \text{sgn} \, \alpha_0) = 0. \]

Obviously, \( \varepsilon \alpha_0 < 0 \) when the pear-shaped solutions occur. The conclusion follows immediately from theorem 2.

5. AN EXAMPLE

In this section we apply the results obtained above to a special case. Specifically, we assume that the mass of an ideal gas enclosed by the deformed membrane is controlled during the inflation (cf. (6)), and that the strain-energy function of the membrane has the properties described below. Recall that the internal pressure needed to maintain a spherical
deformation with radius \( \lambda \) is given by (11). We find it convenient to define the new variable

\[
\xi \equiv \frac{1}{\lambda}.
\]  

Then equation (11) can be rewritten as

\[
p = \xi^2 \tilde{W}'' \left( \frac{1}{\xi} \right).
\]  

We henceforth assume that as \( \lambda \) increases from the undeformed state (\( \lambda = 1 \)), the pressure \( p \) initially increases to a local maximum, and then drops off to attain a local minimum. This property is shared by a large class of strain-energy functions, e.g. those considered in [1, 3, 4, 6]. A typical plot of \( p \) vs \( \xi \) for such a strain-energy function is sketched in Fig. 2. For future reference, we note that there are two points \( \xi^* \) and \( \xi^{**} \) where the tangent of the \( p-\xi \) curve passes through the origin.

Successive differentiation of (37) yields

\[
\frac{dp}{d\xi} = 2\xi \frac{d\tilde{W}''}{d\xi^2} - \frac{d\tilde{W}''}{d\xi}, \quad \frac{d^2p}{d\xi^2} = 2 \frac{d\tilde{W}''}{d\xi} - \frac{2}{\xi} \frac{d\tilde{W}''}{d\xi} + \frac{1}{\xi^2} \frac{d\tilde{W}''}{d\xi}.
\]  

It immediately follows from (36), (37), (38), and Fig. 2 that equation (28) holds at \( \lambda^* = 1/\xi^* \) and \( \lambda^{**} = 1/\xi^{**} \); these are two possible bifurcation points. From (4), (6) and (11), we find that

\[
\varphi^0_{\nu} = \frac{3W_1^0}{2\pi\lambda^3_0}, \quad \varphi^0_{\mu} = \frac{2W_1^0}{\lambda^3_0\mu_0}.
\]  

The substitution of these into (33)_6, using (28) and (38)_2, yields

\[
\alpha_0 = \frac{1}{3\mu} \frac{dp}{d\xi^2} \quad \text{at} \quad \xi^* \quad \text{and} \quad \xi^{**}.
\]  

Observe from Fig. 2 that \( \alpha_0 \) is negative at \( \xi^* \), and positive at \( \xi^{**} \). Assuming \( \beta_0 > 0 \), it then follows from theorem 1 that \( \lambda^* \) and \( \lambda^{**} \) are indeed bifurcation points. The bifurcation diagram is shown schematically in Fig. 3. We recall that the bifurcation parameter \( \mu \) now stands for the mass of the enclosed gas. From (4), (6) and (37), the relationship between \( \mu \) and \( \xi \) for spherical deformations is

\[
\mu = \frac{4\pi}{3k_0} \tilde{W}'' \left( \frac{1}{\xi} \right).
\]
Hence, the bifurcation points $\mu^*$ and $\mu^{**}$ are found by evaluating (39) at $\xi^*$ and $\xi^{**}$, respectively. The $\mu$-axis in Fig. 3 coincides with the spherical solution branch, which is unstable (cf. Corollary 1) between $\mu^*$ and $\mu^{**}$, as indicated by the dotted line. The ordinate $y$ provides a measure of the deviation of the solution from spherical deformations.

Our local bifurcation analysis is valid only in a neighborhood of a bifurcation point, and says nothing about whether the two non-spherical solution branches bifurcating from $\mu^*$ and $\mu^{**}$ are connected. However, Alexander's experimental observations [3] and Haughton's numerical results [6] strongly suggest that it is the case. In the remainder of this paper, we show that the bifurcation branches do form a closed loop when the points $\xi^*$ and $\xi^{**}$ in Fig. 2 are sufficiently close to each other.

Consider a one-parameter family of smooth strain-energy functions $W(\lambda_1, \lambda_2, \tau)$, where $\tau$ is a material parameter, that takes values in a neighborhood of zero. Introducing this parameter into (37), we obtain

$$p(\xi, \tau) = \xi^3 \hat{W}'\left(\frac{1}{\tau}, \tau\right),$$

(40)

where $\hat{W}(\lambda, \tau) \equiv W(\lambda, \lambda, \tau)$. Equation (40) defines a one-parameter family of $p-\xi$ curves, which is assumed to have the following properties: for positive values of $\tau$ the $p-\xi$ curve has the feature as described in Fig. 2, with $\xi^*$ and $\xi^{**}$ corresponding to the two bifurcation points. As $\tau$ decreases, they move toward each other such that $\xi^* = \xi^{**} \equiv \xi_0$ when $\tau$ vanishes. This behavior is characterized by the following conditions

$$p - \frac{\xi}{\xi^2} \frac{\partial p}{\partial \xi} = 0, \quad \xi^2 \frac{\partial^2 p}{\partial \xi^2} = 0, \quad \frac{\partial}{\partial \xi} \left( p - \frac{\xi}{\xi^2} \frac{\partial p}{\partial \xi} \right) < 0,$$

(41)

where all terms are evaluated at $(\xi^*, \tau) = (\xi_0, 0)$. The $p-\xi$ curves for several values of $\tau$ are sketched in Fig. 4.

The equations of equilibrium now take the form

$$G(x, \mu, \tau) = 0,$$

(42)

where $G(x, \mu, \tau)$ is defined analogously to (16), with the additional variable $\tau$ present as an argument of $W$. Recalling (18) and (39), we observe that

$$(x, \mu, \tau) = ((\lambda_0 \cos \theta, \lambda_0 \sin \theta), \mu_0, 0),$$

$$\lambda_0 = \frac{1}{\xi_0}, \quad \mu_0 = \frac{4\pi}{3k\xi_0} \hat{W}'(\lambda_0, 0),$$

(43)
is a spherical solution of (42), about which the linearized equations of equilibrium have a mode one solution (27).

Obviously, our prior analysis carries over to the investigation of (42) near the singular point (43). The problem again reduces to a one-dimensional one

\[ g(\gamma, \tau) = 0. \]  

(44)

All previously calculated derivatives of \( g \) listed in (33) remain unchanged, and other derivatives can be found similarly, for example,

\[ g^{0}_{\gamma \tau} = \frac{4}{3} \left[ \lambda_0 \psi_0 (W_{111}^0 + 3W_{122}^0) - W_{11}^0 + \lambda_0 (W_{11}^0 + W_{12}^0) \right], \]

(45)

where \( g^{0}_{\gamma \tau} \equiv g_{\gamma \tau}(0, 0, 0) \), \( W_{11}^0 \equiv W_1(\lambda_0, \lambda_0, 0) \), etc. Substituting (40) into (41), we find that

\[ W_{11}^0 - \lambda_0 (W_{11}^0 + W_{12}^0) = 0, \]

\[ W_{111}^0 + 3W_{111}^0 = 0, \]

\[ W_{11}^0 - \lambda_0 (W_{11}^0 + W_{12}^0) > 0. \]

(46)

It then follows from (33), (45) and (46) that

\[ g^{0}_{\gamma \tau} < 0, \quad \alpha_0 = g^{0}_{\gamma \tau} = 0. \]

**Theorem 3.** Under the conditions (41) and the assumptions

\[ g^{0}_{\gamma \gamma} \neq 0, \quad g^{0}_{\gamma \tau} \neq 0, \]

\[ y \]

\[ \gamma \]

\[ \tau \]

(51)

**Fig. 5.** Bifurcation diagram of a universal unfolding.
the local bifurcation diagram of (42) is equivalent to the one generated by the normal form

\[ y(e_1 y^2 + e_2 y^2 - \tau) = 0, \]  

(47)

where

\[ e_1 \equiv \text{sgn} \, g_{yy}^0, \quad e_2 \equiv \text{sgn} \, g_{yy}^0. \]

The proof of Theorem 3 can be carried out by singularity theory treated in [7]. In particular, we recognize (47) as a \( \mathbb{Z}_2 \) universal unfolding of \( y(y^2 \pm y^2) = 0 \). For the case \( e_1 = e_2 = 1 \), the bifurcation diagram (47) is graphed in Fig. 5. It is obvious that the intersection of the graph and the plane \( \tau = \text{constant} > 0 \) forms a circle crossed by a straight line from the middle. This shows that the two non-spherical solution branches in Fig. 3 are indeed connected to each other, at least when the two bifurcation points \( \mu^* \) and \( \mu^{**} \) are sufficiently close.

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REFERENCES


APPENDIX: DERIVATION OF (31)–(33)

Let \( P \) be the projection in \( L^2 \) onto the orthogonal complement of \( \psi \):

\[ P \psi = \frac{\psi}{\pi} \left( \int_0^\pi \psi \, d\theta \right) \psi. \]

(41)

Equation (17) can be written as two equivalent equations:

\[ P G(x_0 + y\phi + h, \mu_0 + \gamma) = 0, \]

(42)

\[ \int_0^\pi \psi \cdot G(x_0 + y\phi + h, \mu_0 + \gamma) \, d\theta = 0, \]

where \( y \in \mathbb{R}, \gamma \in \mathbb{R}, \) and \( h : [0, \pi] \rightarrow \mathbb{R}^2 \) is orthogonal to \( \phi \):

\[ \int_0^\pi h \cdot \phi \, d\theta = 0. \]

(43)

Obviously, upon a change of variables \( x = x_0 + y\phi + h, \mu = \mu_0 + \gamma, \) that translates the anticipated bifurcation point \( (x_0, \mu_0) \) to the origin, \( \gamma \) is now the control variable, \( y \) the component of the displacement \( x - x_0 \) in the direction of the kernel \( \phi \), and \( h \) the remaining displacement. In (42), we have projected equation (17) on the co-kernel \( \psi \) and on its orthogonal complement.

The Fréchet derivative of the left-hand side of (42), with respect to \( h \) is a linear mapping from the complement of \( \phi \) onto the complement of \( \psi \), and is therefore invertible at \( (h, \gamma) = (0, 0) \) by Lemma 1. It then follows from the implicit function theorem that one can solve (A2), locally for \( h \):

\[ h = h(y, \gamma). \]

(44)

Substituting (44) into (A2) then leads to (31) with \( g : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) being defined by

\[ g(y, \gamma) = \int_0^\pi \psi \cdot G(x_0 + y\phi + h(y, \gamma), \mu_0 + \gamma) \, d\theta. \]

(45)
Our problem is invariant under reflections about the $x_3$-axis. It is an easy matter to check that the function $G$ defined by (16) satisfies
\[
G\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x, \mu \right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G(x, \mu),
\]
where the volume $V$ is redefined as the absolute value of the right-hand side of (3) so that a reflection leaves $V$ unchanged. This symmetry is subsequently inherited by functions $h$ and $g$. Precisely, $g(y, y)$ is odd in $y$ as stated by (32). Thus, the value of $g_1$, its derivatives with respect to $y$, and its even-order derivatives with respect to $y$ all vanish at $(0, 0)$. Other results in (33) can be obtained by taking derivatives of (A2) and (A5) and using the chain rule:
\[
A_2 h_1^0 = 0, \quad A_0 h_1^0 = -PG_0^0,
\]
\[
g_2^0 = 0, \quad g_2^0 = \int_0^\pi \psi \cdot \left( G_0^0 h_2^0 + G_0^1 (\phi + h_1^0) \right) d\theta. \tag{A6}
\]
Hence we have used the fact that $\phi$ is a null vector of $A_0$ and $\psi$ is orthogonal to the range of $A_0$. By using the solution of (30) given in the proof of Lemma 1, one can solve (A6)$_{1,2}$ for $h_2^0$ and $h_1^0$. Substituting them into (A6)$_{1,2}$ then yields (33)$_{1,2}$. 